The effects of variation in management objectives on responses to forest diseases under uncertainty

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Abstract

The real options approach provides a powerful tool for determining the optimal time at which to adopt disease control measures given that there is uncertainty about the future spread of an invading pest or pathogen. Previous studies have considered the timing of control from the point of view of a central planner. However, decisions regarding the deployment of control measures are typically taken by individual forest managers, who may have widely differing objectives. In this article we investigate how management objectives impact the optimal timing of control measures given uncertainty in disease spread. Our results show that differences in management objectives can lead managers to act at different times, and potentially never adopt disease control. In particular, these differences in the timing of disease control for diverse types of managers becomes more significant if the disease impacts the range of benefits from the forest to different extents. This creates tensions at the landscape scale where there are managers with divergent objectives due to the transferable externality (the disease). Our results have important implications for national decision making bodies and suggest that incentives may need to be targeted at specific groups to ensure a coherent response to disease control.

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There has been a significant rise in invasive pathogens and pests within forests in recent years, and the problem is set to worsen in the future, largely due to climate change and changing trade patterns (Hanley and MacPherson 2016; Sturrock et al. 2011; Freer-Smith and Webber 2015). Such pathogens and pests can cause significant damage (often mortality) to trees leading to negative economic and ecological impacts on forests and so pose an important threat to the British forestry industry, and to public goods such as recreation and biodiversity conservation that forests provide.

When a new pest or pathogen arrives, a key choice facing a forest owner is whether or not to adopt control measures to reduce the spread of the pest or pathogen. The owner must weigh up the costs of control against the potential reduction to the benefits of their forest as a result of pest or pathogen damage. However, these future benefits are uncertain due to the uncertainty in the future progression of the pest or pathogen as a result of environmental fluctuations such as climatic conditions, evolution within the pathogen species, and scientific uncertainty about how the disease is transmitted and about the effectiveness of treatment options. Furthermore, owners differ in the benefits they obtain from the forest depending on their given objectives. So while it may be in one owner's interest to adopt control, the same is not necessarily to true for another owner with different objectives for their forest. A key aim of this article it to bring these two elements together so as to investigate how the interaction between future uncertainty in disease spread and forest management objectives affect when (if ever) it is optimal for an individual forest manager to adopt control measures to mitigate damage due to invasive pathogens or pests.

A major economic challenge in controlling invasive pests and pathogens is that such measures involve both sunk and on-going costs. Sunk costs represent irreversible losses to the land manager. This combined with the uncertainty in the future progress of disease means there can be value in waiting to learn more before initiating costly control measures (J

Saphores 2000; Sims and Finnoff 2012; Sims and Finnoff 2013; Ndeffo Mbah et al. 2010; Marten and Moore 2011). Previous studies have shown this using a real options approach, since it provides a powerful way to analyse the joint effects of uncertainty and learning, in the context of irreversibility. By viewing disease control as an option that can be exercised to reduce damage by a pest or pathogen to the host species, the real options approach provides a threshold in the proportion of infected area at which it is optimal to deploy control immediately.

Saphores (2000) first applied the real options approach to determine the impact of uncertainty in future area infected by a pest on the optimal timing of pesticide application. Varying the level of uncertainty, Saphores (2000) showed that greater uncertainty in future infected area increases the threshold at which it is optimal to spray. This gives rise to the "wait-and-see" approach to dealing with invasive pests or pathogens: when there is uncertainty in the future dynamics of area infected by a pest or pathogen, there is value in waiting to learn more before investing in control. Saphores (2000) assumes that the application of pesticide is irreversible, that is once pesticide application is initiated it can never be cancelled. However, in reality this is usually not the case since such measures are on-going and can be stopped at any time. Indeed, Sims et al. (2016) argue that the only example of a truly irreversible control measure, i.e. one that cannot be stopped at some point in the future once it has been initiated, is the release of a biological control agent. Therefore, Sims & Finnoff (2013) consider the implications of the reversibility of the control measure on how long a regulator should wait to take action. They find that if control measures are partly reversible (for example trade bans), then it is optimal to act earlier than if measures are completely irreversible. This shows that how long it is optimal to "wait and see" depends not only on the future uncertainty in infected area but also on the nature of actions which can be taken, i.e. how reversible the control measures are.

Traditionally, the uncertainty in the spread of the pest or pathogen is assumed to follow a Geometric Brownian Motion (GBM) (Saphores 2000). The advantage of such an assumption is that GBM is well understood and allows for closed-form solutions to the real options problem. However, GBM is unbounded above and so it does not respect the upper boundary in the infected area which arises due to the limited number of available hosts within a given spatial domain. Sims & Finnoff (2012) consider the impact of introducing a spatial boundary on the timing of control, by incorporating an exogenous upper boundary into the decision problem. This is to prevent the threshold at which to act rising above the maximum area that can be infected. They find that spatial scale can affect the timing and stringency of control strategies, and so incorporating an upper bound is important in planning measures to minimise losses from pest or pathogen damage (Sims and Finnoff 2012).

Other authors have tried to incorporate the limiting nature of pest or pathogen spread by using more realistic processes to describe the uncertainty in future infected area. Ndeffo Mbah et al. (2010) incorporate a logistic-type term into the drift coefficient of the stochastic differential equation (SDE) describing the increase in area infected by the pest or pathogen, so the mean growth of the process is limited by a parameter that represents the maximum area that can be infected. They find that the new SDE leads to qualitatively different behaviour in the relationship between the optimal timing of control adoption and uncertainty. While the SDE used by Ndeffo Mbah et al. (2010) more realistically captures the logistic nature of growth in area infected by the pest or pathogen, such a process does not respect the upper boundary in infected area. Dangerfield et al. (2016) show that incorporating a logistic-type term into both the drift and diffusion coefficients of the SDE describing growth in infected area ensures that the upper boundary is endogenously incorporated into the decision problem. Formulating the dynamics using this more realistic stochastic process leads to control being deployed earlier, which has important implications for decision makers such as forest managers or government agencies.

In previous studies, the optimal timing of control is typically considered from the point of view of a central planner (Sims and Finnoff 2013; Sims and Finnoff 2012; Marten and Moore 2011). However, since areas of forest can be privately owned, for example in Britain it is estimated that around 70% of forests are privately owned (Fitter 1980), whether or not to adopt control measures will be decided by the individual owners, rather than a central planner. Furthermore, the way in which an owner manages their forest will depend on a specific set of forestry objectives, which can vary significantly between different owners. Urquhart & Courtney (2011) identified six distinct groups of forest owners in England based on common objectives, motivations and attitudes within each group. For example, Urquhart & Courtney (2011) defined the 'investor' group as those who prioritise timber production and investment opportunities in their forestry objectives, and do not manage their forests for non-timber benefits such as wildlife conservation or recreation. On the other hand, the 'conservationist' group were motivated to manage their forests to conserve biodiversity and do not consider timber production as a significant output (Urquhart and Courtney 2011). This heterogeneity in the priorities of different forest owners means they place different weightings on the individual benefits obtained from the forest. Combined with this is the fact that pests and pathogens do not affect the different benefits of the forest uniformly, and so there is inconsistency in the level of effect a given pest or pathogen has on a particular benefit. For example, dothistroma needle blight causes premature needle loss on pine trees, which leads to significant reductions in tree growth and so can result in substantial losses in timber value over a rotation period[1]. On the other hand, this reduction in growth has a lower impact on biodiversity and amenity value. In such situations there is less of an incentive for

[1] http://www.forestry.gov.uk/pdf/fcrn002.pdf/$FILE/fcrn002.pdf

an owner within the conservationist group to control against the pathogen or pest as compared with an investor whose priority is timber production. Differences in the objective function explicitly or implicitly used by each forest owner are therefore likely to affect how they respond to a particular pest or pathogen risk, and thus how that pest or pathogen spreads across a landscape.

In this article we investigate how the objectives of a forest owner affect how long they will optimally wait before adopting a control measure, given that there is uncertainty over the future progress of the pest or pathogen. Using a real options approach to incorporate uncertainty into the decision process, we explore how varying the function that describes the value of the forest (the owner's objective function) influences the threshold at which control should be adopted immediately. This provides insight into how variations in the different benefits obtained from a forest will affect an individual forest owner's optimal time to control. These benefits could reflect the utility which a forest owner gets from both timber and non-timber attributes, and/or a Payment for Ecosystem Services which they receive for producing public good-type benefits along with the financial benefits of timber production. We model the forest owner as taking decisions over control measures in their forest independently of other forest owners, in the sense that expectations over the actions of others do not enter into their decision-making process.

We find that while the heterogeneity in forest management objectives leads to differences in the region of infected area at which control should be adopted, these differences are small when the pest or pathogen affects the timber and non-timber benefits of the forest in a uniform way. Therefore, our results suggest that the diversity in forest management objectives alone does not lead to significant variation in the timing of pest or pahtogen control measures between different managers. On the other-hand when the heterogeneity of management objectives is combined with differences in the reduction of timber and non-timber benefits then the discrepancies between the control strategies of dissimilar managers becomes more significant. Indeed it may be optimal for one manager never to adopt control, while the other should adopt control as soon as the proportion of infected area exceeds some threshold level. In such situations, we show that the discrepancy in the control strategies of different managers can be reduced through subsidy schemes that reduce the costs of control. Our results have important implications for local and national decision making bodies, such as the UK Government Department for Environment Food and Rural Affairs (DEFRA), the Forestry Commission in England and Scotland and Natural Resources Wales, who seek to achieve reductions in pest or pathogen spread at a larger spatial scale than an individual forest.

Methods

In this article, we use terminology typically associated with an invasive pathogen rather than pest, and so we refer to trees as being infected or diseased, rather than invaded. This is for ease of writing, but we note that the model frameworks described here apply equally well to invasive pests, such as oak processionary moth and oriental gall wasp, two pests that have been found in England in recent years.

Value of the Forest in the Absence of Disease

Consider an area of even-aged forest composed of a single species that is of size L hectares. We assume the value of the forest after a fixed period of time T years, where T is an exogenous time that could, for example, represent the length of a pre-determined rotation period, to be composed of two parts: a single payment that is received at the final time T, which represents the net return from selling the timber, $M(L)$, and an annual payment that characterises the value obtained from a flow of non-timber benefits, $S(L)$. These non-timber benefits could for example describe amenity, recreation or biodiversity values for which either (i) the owner derives utility or (ii) for which they obtain a payment from a third party such as the government. Therefore, the present value of benefits from the forest, in the absence of disease, is given by the following equation,

$$V_{DF}(T) = M(L)e^{-rT} + \int_0^T S(L)\, e^{-rt} dt, \tag{1}$$

where r is the discount rate.
We assume that functions $M(L)$ and $S(L)$ take the following forms,

$$M(L) = pL \tag{2}$$

$$S(L) = bL \tag{3}$$

where p is the net return per hectare from timber sold at the end of the rotation and b is the value per hectare of annual non-timber benefits.

Value of Forest in the Presence of Disease

Consider the outbreak of a disease within the forest of interest. We assume that the future progress of area infected is uncertain due to the variability in infection transmission as a result of external factors such as climatic conditions. Therefore, the proportion of area infected over time (I) changes according to the following stochastic differential equation SDE, which we term the logistic SDE, (Dangerfield et al. 2016),

$$dI = \beta I(1 - I)dt + \sigma I(1 - I)dW, \tag{4}$$

where β is the transmission rate and σ is the level of uncertainty. Since the noise term (given by the second term in equation (4) represents the fluctuations in the transmission parameter β due to external factors, we assume that σ scales with β and so $\sigma = \beta \times F$. The constant F measures the magnitude of the fluctuations associated with the transmission parameter β

(Keeling and Rohani 2008). Formulating the uncertainty parameter in this way ensures that when the transmission rate falls to 0 (i.e. when there is no transmission), the uncertainty in disease spread also falls to 0. We describe the future uncertainty in infected area using the logistic SDE since it is more realistic than GBM which is typically used in the literature (J Saphores 2000; Sims and Finnoff 2013), and so ensures applicability of our results to real-world epidemics (Dangerfield et al. 2016).

We assume that both the timber and non-timber benefits from the forest are reduced by disease, and so infected timber value is decreased by a factor $0 \leq \rho \leq 1$ (e.g. because of lower timber volume as a result of reduction in tree growth rate) and similarly annual non-timber benefits from infected trees are lowered by a factor $0 \leq \varphi \leq 1$. We assume that ρ and φ are independent and consider the impact of a range of different combinations of ρ and φ on the optimal timing of control. The value of the forest, per hectare, in the presence of disease is given by

$$V_D(I,T)\times L = E\left[p(1+(\rho-1)I(T))e^{-rT} + L\int_0^T b(1+(\varphi-1)I(t))\, e^{-rt}dt\right] \tag{5}$$

where $I(t)$ is the proportion of the forest area (L) that is infected at time t. Note that this is the expected value of the forest (E in equation (5) represents the expectation), since $V_D(I,T)$ is now stochastic, due to the uncertainty in the level of infection.

Optimal Timing of Disease Control

Consider a control policy that reduces the rate at which the disease spreads by a factor $0 \leq \omega \leq 1$, so the transmission rate after a control option is implemented is $\beta_A = \beta\times\omega$. Since the volatility ($\sigma$) is a function of β, the volatility will also be reduced after the initiation of control measures, that is $\sigma_A = \beta_A \times F$. Examples of such control measures would be restrictions on the movement of people and vehicles, increased biosecurity measures such as the washing of footwear and boots, the removal of weeds and debris from the forest understorey, or chemical spraying treatments that reduce the susceptibility of trees. Routine thinning of the forest is also advised by the Forestry Commission to reduce the transmission of some diseases[2], however our model does not accurately encompass this since we do not directly account for the removal of trees (and thus changes in the density of individuals) within the epidemic model. We thus omit thinning from the list of control options which the model describes.

We assume that control can be adopted for a fixed cost of K_A per hectare, and that there is a yearly maintenance cost of m_A per hectare to continue control. Fixed costs are non-recoverable, and represent a one-off upfront cost that could, for example, be the initial investment in specialist equipment, or the cost of time taken to initiate the control policy. The yearly cost represents the annual payment needed to continue the control programme, for example this could be the yearly payment to contractors to remove weeds or apply chemical sprays or alternatively the ongoing cost of increased biosecurity measures. Since many control measures, with exceptions such as the release of a biological control agent, can be cancelled at some point in the future, we assume that the control measure is temporary, and so we can consider the decision to invest in control as reversible. Therefore, if control is currently being adopted then such a programme can be cancelled at some point in the future, and when this occurs we consider that a portion of the initial fixed cost is recouped, which is denoted by $K_C = \alpha K_A$ (where $\alpha \in [0,1]$). This could, for example, represent the sale of specialist equipment. In the forestry sector in the UK, many control measures, such as weeding and thinning, are currently outsourced to contractors and so it is unlikely that any

[2] http://www.forestry.gov.uk/pdf/fcrn002.pdf/$FILE/fcrn002.pdf

costs will be recouped by the forest owner upon the cancellation of control measures. Therefore, in all our analysis we take $\alpha = 0$, however we keep this parameter in the formulation of the decision problem to ensure the generality of our method to other problems.

Let $W_N(I,t)$ be the value of the forest when control is not adopted. It comprises two parts: the discounted expected value of the forest if nothing is done and the value of the option to postpone adoption of control. This option value arises since the future uncertainty in the proportion of infected area means that there is an opportunity cost of applying control immediately, rather than waiting to see what happens in the future. Similarly, if control measures are currently being applied then there will be an opportunity cost associated with cancelling them now rather than waiting. Therefore the value of the forest when control is being adopted, $W_A(I,t)$, will be the discounted expected value of the forest obtained when control is applied plus the value of the option to postpone cancellation.

If there are currently no control measures in place then control should be adopted as soon as the area infected reaches I_A, which we term the ***adoption threshold***. At I_A the following two boundary conditions are satisfied:

$$W_N(I_A, t_A) = W_A(I_A, t_A) - K_A \tag{6}$$

$$\frac{\partial W_N}{\partial I}(I_A, t_A) = \frac{\partial W_A}{\partial I}(I_A, t_A). \tag{7}$$

The first condition is called the value matching condition and ensures that the payoff from adopting control immediately is equal to the payoff from not adopting control. The second condition is called the smooth pasting condition and requires $W_N(I,t)$ and $W_A(I,t)$ to meet tangentially at I_A to ensure the optimality of I_A (see (Dixit and Pindyck 1994) for further discussion).

Similarly if control measures are currently being adopted then control should be cancelled as soon as the area infected reaches I_C, which we term the ***cancellation threshold***. At I_C the following two boundary conditions are satisfied:

$$W_A(I_C, t_C) = W_N(I_C, t_C) + K_C \tag{8}$$

$$\frac{\partial W_A}{\partial I}(I_C, t_C) = \frac{\partial W_N}{\partial I}(I_C, t_C). \tag{9}$$

Once again, the first condition ensures that the payoff from cancelling control immediately is equal to the payoff from not cancelling control while the smooth pasting condition (equation (9)) ensures optimality of I_C (Dixit and Pindyck 1994).

Following the standard dynamic programming approach, the value of the forest when control is not adopted, $W_N(I,t)$, and when it is adopted, $W_A(I,t)$, will satisfy the following partial differential equations (PDEs)

$$\frac{\partial W_N}{\partial t} + \sigma^2 I^2 (1-I)^2 \frac{\partial^2 W_N}{\partial I^2} + \beta I (1-I) \frac{\partial W_N}{\partial I} - rW_N + b + b(\varphi - 1)I = 0, \tag{10}$$

$$\frac{\partial W_A}{\partial t} + \sigma_A{}^2 I^2 (1-I)^2 \frac{\partial^2 W_A}{\partial I^2} + \beta_A I (1-I) \frac{\partial W_A}{\partial I} - rW_A - m_A + b + b(\varphi - 1)I = 0, \tag{11}$$

subject to the boundary conditions given by equations (6) to (9) and terminal conditions

$$W_N(I,T) = p(\rho - 1)I(T) + p, \tag{12}$$

$$W_A(I,T) = p(\rho - 1)I(T) + p. \tag{13}$$

Since the boundary conditions are specified at a point that is yet to be determined, this system is a free-boundary problem. Solving equations (10) and (11) provides the adoption and cancellation thresholds, which will be functions of time since we consider the timing of control over a finite time horizon. In this article we are primarily interested in the optimal timing of control at the beginning of the time horizon of interest, that is $I_A(0)$ and $I_C(0)$. Therefore, unless specified we use I_A and I_C to denote the adoption and cancelation thresholds at time *t = 0*.

Due to the logistic nature of both the drift and diffusion terms in the logistic SDE, it is not possible to obtain closed-form solutions to this problem. Therefore we solve the free boundary problem given by equations (6) to (13) numerically using the Euler method in MATLAB (Wilmott, Howison, and Dewynne 1995). Further details are given in the Appendix.

Results

This is the first time that the logistic equation has been used within an investment decision problem that is reversible. Therefore, we initially explore the behaviour of the adoption and cancellation thresholds for the decision problem outlined in the previous section.

Adoption and Cancellation Thresholds

We find that there exist two adoption thresholds, I_A^L and I_A^U, and similarly two cancellation thresholds I_C^L and I_C^U (Table 2 and Figures 1 a and b). When no control measures are currently in place, our results show that it is optimal to apply control immediately providing that the level of infection lies within the two adoption thresholds, that is providing $I_A^L \leq I \leq I_A^U$. We term this range of I values the ***adoption region***. If the area currently infected, I, is too small, the benefits of control do not outweigh the costs and so control should not be adopted until the proportion of infected area is large enough. Similarly when the proportion of area infected is close to 1 there will be little benefit from adopting control measures, as most of the forest is infected. Such an upper threshold may be exceeded even at the initial time if the damage remains undetected, which could occur, for example, if insufficient resources are devoted to surveillance efforts.

If control is currently being adopted then our results suggest that a forest manager should cancel control and go back to doing nothing as soon as the level of infection drops below or above the cancellation thresholds, that is when $I \leq I_C^L$ or $I_C^U \leq I$. We term this the ***cancellation region*** and we note that this region is disjoint. Similarly to the adoption thresholds, the upper and lower thresholds arise since at very low or high levels of infection the benefits from control no longer offset the costs, and so control measures should be cancelled immediately.

The existence of two adoption and cancellation thresholds is in contrast to traditional results within the investment and real options literature, where such decision problems typically lead to a single adoption threshold and single cancellation threshold (Dixit and Pindyck 1994; Saphores 2000). Conventionally Geometric Brownian Motion (GBM) is used to describe the future uncertainty in infected area (Saphores 2000; Saphores and Shogren 2005) and such a process is unbounded. In contrast the logistic SDE (equation (4)) used here is bounded above by 1, which represents the total area that can be infected. The incorporation of the spatial scale that is implicit in the spread of disease endogenously into the decision problem results in both the value of the forest if you never adopt control, and the value of the forest if control is adopted being convex, Figure 2. The difference in the convexity of the two value functions (Figure 2) arises because of the reduction in the transmission rate, β, after control is adopted. The benefit of control is that it reduces the loss in value of the forest as the level of infection increases, since the impact of control is to slow the spread of disease. Near the boundaries, 0 and 1, this reduction does not outweigh the costs of control (shown on Figure 2 by comparison of the red solid line and the dashed green line) and so control should not be adopted. Therefore, the upper boundaries arise as a result of the spatial upper boundary of the system. In particular, since the upper thresholds exist in the absence of uncertainty (Figure 1 c and d), suggests that future uncertainty is not the driving factor for the presence of these upper thresholds. This is similar to the findings of Sims and Finnoff (2012) where the authors find the existence of an upper threshold when they enforce an upper boundary on GBM representing the maximum area that can be infected.

The impact of uncertainty is to increase the lower adoption threshold, I_A^L, while decreasing the upper threshold, I_A^U, so the net effect is to reduce the size of the adoption region (Figure 1a and c). Similarly, the lower cancellation threshold, I_C^L, increases with increasing uncertainty, while the upper threshold decreases, I_C^U (Figure 1b and d). When future uncertainty is taken into account, the area infected can both increase and decrease over time. This gives rise to an opportunity cost of adopting control now rather than waiting to see how the infection progresses within the forest. This opportunity cost is termed the option value within the real options literature and characterises the benefit from waiting before adopting control, which arises due to the uncertainty in the future progression of infection. Therefore, the option value increases the benefit of waiting to adopt control and so the lower adoption threshold increases. Analogously the same is true for the lower cancellation threshold.

As uncertainty increases, both the value of the forest when adopting and the value of the forest when not adopting control increase as a result of the option values to adopt or cancel respectively. However, the increase in the value of the forest when not adopting control is greater than the increase in the value of the forest when adopting. This difference arises since we assume that the magnitude of the volatility term in equation 4 depends on transmission parameter β. Since β decreases after control is adopted, for a fixed uncertainty constant F the result is that the magnitude of the volatility term will also decrease after control adoption and so the increase in the value of the forest as a result of adopting will be less. This difference in the increase in the value of the forest when adopting and not adopting is what causes the upper threshold to decrease as uncertainty increases. Again this is also why the upper cancellation threshold decreases with increasing uncertainty.

Managing for Timber versus Non-timber Objectives

Since the main aim of this article is to understand the impact of a forest owner's objectives on the timing of control adoption, we focus on the adoption and cancellation thresholds for two forest owners with divergent objectives for their forest.

So far we have discussed the different objectives of the forest owner, not the manager, since the "owner" is the person who sets the objectives for a forest while the "manager" is the person who maintains the forest (this could be the owner themselves or someone acting as an agent for the owner). In what follows we consider the decision of whether or not to adopt control, which will ultimately be made by the forest manager, to be in line with the objectives set out by the owner. Therefore, to avoid confusion between these two terms, from here on we use the terminology of manager rather than owner, and so when we refer to the forest managers objectives, these will be the objectives set out by the owner.

Consider two forest managers who are managing for different objectives: the first manages a forest solely for the timber benefits ($b = 0$ in equation (5)), while the second manages a forest solely for the non-timber benefits ($p = 0$ in equation (5)). We assume that both managers take decisions with regards to disease control independently of the other's and so they do not take into account expectations of the others actions in their decision-making process. To ensure a fair comparison between the two, we set b so that the value of the forest in the absence of disease for the non-timber manager is the same as for the timber manager when $p = 1$. Independently varying the reduction in timber (ρ) and non-timber benefits (φ) as a result of disease, we initially examine the impact of increasing disease damage on the adoption and cancellation regions of control for the two managers.

Impact of increasing disease damage on timing of control

We find that for both forest managers, when the damage due to infected trees is very low (so ρ or φ are close to one), the adoption thresholds do not exist and so it is never optimal to apply control. This is denoted in Figure 3 by the '*never adopt control*' region. The advantage of adopting control is the reduction in the speed of infection spread and resulting increase in the timber and non-timber benefits as more trees remain healthy for longer. If infection reduces the timber benefits by very little then the increase in the value of the forest when adopting control versus not adopting will be small. Indeed, in the never-adopt region, the reduction in benefits from the forest due to slower disease spread leads to an increase in the value of the forest when adopting that does not outweigh the costs of control. Therefore the benefits of control do not justify the costs and so there is no level of infection at which control measures should be undertaken. As the damage due to disease increases (ρ or φ move further from 1), the difference in the value of the forest when adopting control versus not adopting increases. This leads to the appearance of two adoption thresholds and so there is a region in which it is optimal to apply control. As ρ or φ become closer to zero, the benefit from adopting control rises and so the adoption region becomes larger (Figure 3). Similarly the size of the cancellation region becomes smaller (Figure 4). We term the level of damage at which the optimal strategy switches from 'never control' to 'control within the adoption region' the **strategy switch point**.

Providing the disease affects both timber and non-timber benefits to the same extent, that is $\rho = \varphi$, in the region where the adoption thresholds exist we find that the lower adoption threshold for the timber-objective manager is always smaller than for the non-timber-objective manager, Figure 3a. The same is also true for the upper thresholds. Furthermore, the strategy switch point for the non-timber manager is higher than for the timber manager and so the reduction in non-timber benefits due to disease must be greater before it is optimal for the non-timber manager to adopt control. However, the difference between the strategy switch points is very small when $\rho = \varphi$ and so the region where it is optimal for the timber manager to control within the adoption region, while the non-timber manager should never control, is very small. While we find that there are differences in the size and positioning of adoption regions between timber and non-timber managers, for diseases for which $\rho = \varphi$ there is a large overlap in the region in which both managers should adopt. Similarly, the cancellation regions for the two managers are very similar (Figure 4a). Therefore, in these situations differences in management objectives will lead to similar control strategies for the two different types of managers.

The reduction in timber and non-timber benefits will not always be the same and will depend on the impact of a given disease on the trees. We find that when $\rho \neq \varphi$, the differences between the adoption regions for the timber and non-timber managers becomes greater and so more significant. In Figure 3b we show the adoption regions for both types of manager, in the case where $\rho = 1 - \varphi$, so along the x-axis reduction in timber benefits is increasing ($1 - \rho$ is increasing) while the reduction in non-timber benefits is decreasing ($1 - \varphi$ is decreasing). When the reduction to timber or non-timber benefits is in an intermediate range there is an overlap in the adoption regions between the optimal strategies of the timber and non-timber managers. However this overlap is significantly smaller than when $\rho = \varphi$. Similarly the cancellation regions for the two managers begin to diverge as ρ and φ become more disparate, Figure 4b. At the extreme values of the reduction in benefits, the difference between the optimal strategy for the timber and non-timber managers is greatest. In particular, depending on the extent to which ρ and φ differ, it may be optimal for the non-timber manager never to adopt control, while the timber manager should control when the infected area lies within the adoption region (and vice-versa). Such cases are summarised in the bottom two rows of Table 3, along with examples of diseases where such a situation arises.

Our results show that when the reduction to timber and non-timber benefits is identical there is little difference between the optimal control strategies of two managers with different objectives. This implies that for diseases such as chalara ash dieback, which causes significant reductions in both timber and non-timber benefits, the range of management objectives between different owners will not lead to a heterogeneous approach to the adoption of disease control measures at the landscape scale. On the other hand when the impact of disease on the different benefits of the forest are disparate, our results show the contrasting objectives between forest managers will lead to significant dissimilarities in their optimal disease management strategies. This is important for the control of disease at the landscape scale since infection can spread from one forest to another (i.e. the spread of infection does not respect land ownership boundaries). Therefore if, for example, the non-timber manager never controls the disease, the benefits of control may be reduced for a neighbouring timber manager, as a result of an increase in the level of infection pressure from the non-timber manager's forest due to no disease control measures.

Impact of Epidemiological Parameters on Adoption Thresholds

When $\rho = \varphi$ we consider how the epidemiological parameters, namely the transmission rate, β, the reduction in spread, ω, and the uncertainty parameter, F, affect the adoption thresholds for the timber and non-timber manager.

As the transmission rate or reduction in the transmission rate due to control increase, the upper and lower adoption thresholds decrease for both the timber and the non-timber managers, Figure (5a and b). The decrease in thresholds is much greater for the timber manager than the non-timber manager. Therefore, when the disease is fast spreading (β high) or the control measure is not very effective (ω high) the difference between the adoption regions for the timber and non-timber managers will be greatest. As uncertainty increases, the upper thresholds decrease while the lower thresholds increase for both the timber and non-timber managers and so the overall size of the adoption region becomes smaller. Once again, the reduction in the size of the adoption region is greatest for the timber manager and so the difference in the timing of control measures between the two managers will be greatest when the uncertainty is large.

Subsidies to align disease control strategies of different forest managers

Consider a disease that has a high impact on timber benefits ($\rho = 0.1$) but little effect on non-timber benefits ($\varphi = 0.9$). In this situation it is optimal for the timber manager to adopt control immediately, providing that the proportion of infected area is within the region [0.004, 0.8] while the non-timber manager should never adopt control. We consider the impact of two different subsidy schemes aimed at the non-timber manager: the first pays out a one-off fixed amount when control is initially initiated while the second pays out a yearly subsidy for the whole period over which control is adopted. The first scheme essentially reduces the fixed cost of adopting control, while the second decreases the yearly maintenance costs that a manager incurs whilst adopting control measures.

Figures 6 and 7 show the adoption and cancellation regions respectively for the non-timber manager as the reduction in fixed costs, that is subsidy scheme 1 (top), and the reduction in yearly costs, that is subsidy scheme 2 (bottom), increase. Also shown are the adoption and cancellation thresholds for the timber manager. Note that we assume that both subsidy schemes are only targeted at non-timber managers, that is the timber managers' costs remain fixed at the baseline values.

We find that both subsidy schemes switch the optimal strategy for the non-timber manager from one where they should never adopt control to one where control should be adopted, providing that the level of subsidy is large enough. In particular we find that the fixed costs of control must be reduced by a greater percentage than the yearly costs in order for the adoption thresholds to exist. This suggests that the adoption thresholds are more sensitive to the yearly maintenance costs (m_A) than the one-off fixed cost (K_A). Furthermore, even if all fixed costs are removed, the size of the adoption region for the non-timber manager is significantly smaller than for the timber manager (Figure 6a). On the other-hand, if the subsidy payments in scheme 2 are large enough to remove all yearly maintenance costs, then the size of the adoption region for the non-timber manager is very similar to that for the timber manager (Figure 6b).

While the adoption thresholds behave in quantitatively the same way for both subsidy schemes, this is not the case for the cancellation thresholds. For subsidy scheme 1, the upper cancellation threshold decreases, while the lower threshold increases as the reduction in fixed costs increases, Figure 7a. Therefore, the overall effect is to increase the size of the cancellation region as the reduction in costs increases. In particular, when fixed costs are completely eliminated the adoption and cancellation thresholds are identical, compare Figures 6a and 7a. This suggests that while there is a region over which it is optimal for the non-timber manager to adopt control, they will cancel control almost immediately upon adoption. On the other-hand, for subsidy scheme 2, the cancellation thresholds behave in the same way as the adoption thresholds, and so the size of the cancellation region decreases as yearly costs are reduced, Figure 7b. Indeed if subsidy payments are large enough to remove all yearly costs there is no longer a cancellation region. Therefore, once control has been adopted, the non-timber manager will not cancel control.

Our results, suggest that subsidy scheme 2, where yearly maintenance costs are reduced, is more effective at aligning the control strategies of the two different types of manager than subsidy scheme 1 which only reduces the initial fixed cost of control adoption. Furthermore, under subsidy scheme 2 the non-timber manager is more likely to continue with control measures after initial adoption compared with subsidy scheme 1. Therefore, reducing on-going costs, rather than on-off fixed costs, also ensures control measures will be sustained in the long term.

Conclusions

There is a wide range of different management objectives amongst forests in the UK, as in other countries. Understanding how this heterogeneity affects the disease control strategy adopted by forest managers, and in particular when they will initiate control measures, is therefore important for national decision-making institutions, such as DEFRA and the Forestry Commission, whose aim is to minimise the spread of disease at the national, rather than at the individual forest, level.

In this article we use a real options approach to investigate how forest management objectives affect when (if ever) it is optimal for the manager of a forest to adopt control measures to reduce damage due to disease, given that there is uncertainty in the future spread of infection. Using the logistic SDE to describe the uncertainty in infection spread, unlike previous studies (Saphores 2000; Saphores and Shogren 2005; Dixit and Pindyck 1994) which assume GBM, we find that there exist two adoption thresholds, and similarly two cancellation thresholds. The upper thresholds arise as a result of the bounded nature of the logistic SDE, which means that when the level of infection is very high there is little to be gained from applying control since most of the forest is infected, and so control should not be applied. Therefore, our results show that control should only be applied when the proportion of infected area lies between these two thresholds, and we term this interval the adoption region. The existence of two adoption thresholds has important implications for decision makers as it suggests that, as well as it being advantageous to wait when the area infected is low, it is also beneficial not to adopt control measures if the area infected is high since the benefits of control do not outweigh the costs.

We compared the adoption region for two forest managers with divergent objectives: the first manages the forest for the timber benefits only while the second manages the forest for the non-timber benefits only. Independently varying the reduction in timber and non-timber benefits as a result of disease we find that when this reduction is small the adoption thresholds do not exist and so control should never be adopted. As the reduction in benefits increases, the size of the adoption region also increases since there is more to be gained from adopting control measures. This is the case for both the timber manager and the non-timber manager.

When the disease reduces timber and non-timber benefits to the same degree, while the adoption thresholds for the non-timber manager are greater than for the timber manager, there is a large overlap between the two adoption regions. Therefore, while our results show that it is optimal for the timber manager to adopt control earlier than the non-timber manager, the difference between the two managers will be small. For policy makers, this implies that in such situations the diversity in management objectives at the landscape scale will not lead to significant differences in the disease control strategy of diverse forest managers. Therefore, it is more likely that a uniform approach to disease management will be achieved without the need for external intervention. However, discrepancies in the adoption regions for the two managers become greater when the disease is fast spreading (β large), the control measure is not very effective (ω large), or the future uncertainty is large (F large).

On the other-hand, if the reduction in timber and non-timber benefits is divergent then there can be significant differences between the timing of control measures for different types of manager. Indeed it can lead to it being optimal for some managers to adopt control while for others it is never optimal to adopt control. Therefore, if the disease has differing impacts on timber and non-timber benefits, for example dothistroma needle blight, the diversity of forest management objectives creates tension in landscapes with multiple owner types, due to a transferable externality (the disease). This has significant implications for policy makers and suggests that they will need to consider other ways in which to incentivise owners whose objectives mean that it is less desirable for them to adopt control in the management of their forest.

Our results have important implications for policy makers since they show that the diversity in management objectives alone does not lead to significantly different disease management strategies between different manager types, but that it is a combination of the diversity in management objectives and the way in which the disease affects these benefits that will determine how uniform the adoption of control measures is at the landscape scale. Hence it is important for decision makers to consider both these factors to ensure homogenous adoption of control across the whole landscape.

When the impacts of a tree disease on the different benefits of a forest are divergent, we find that subsidy schemes that reduce either the fixed cost or yearly maintenance cost can reduce this discrepancy in the timing of disease control measures between different types of forest manager. For a subsidy scheme that only reduces the initial fixed cost, this discrepancy is not eliminated and a significant difference in the adoption region for the two managers remains when all fixed costs are removed. Furthermore, while under such a subsidy scheme the non-timber manager will adopt control within a certain region, it is unlikely that such control measures will be sustained in the long term. This is since the cancellation region becomes larger with increasing subsidy payments (decreasing fixed costs) and when fixed costs are eliminated then the cancellation and adoption thresholds coincide.

On the other-hand, a subsidy scheme that reduces yearly costs ensures that the difference in the timing of control is virtually eliminated, providing the subsidy payments are large enough. Furthermore, control measures are more likely to be continued after initial adoption, since the size of the cancellation region decreases as the subsidy payments increase (yearly costs decrease). Indeed, if yearly costs are completely eliminated then once the non-timber manager has adopted control, they will continue with such measures indefinitely. This has implications for policy makers on the performance of different types of subsidy schemes on aligning the disease control strategies of different types of managers when the disease affects different timber benefits in diverse ways. In particular, our results suggests that subsidy schemes which reduce yearly maintenance costs will be more effective at ensuring that different managers deploy control measures at the same time.

The main aim of this article is to investigate the effect of heterogeneity in management objectives on the timing of control, rather than strategic interactions between forest managers at the landscape scale. In particular, we have assumed that each forest manager makes decisions independently of neighbouring forest managers. However, since the decision of a manager to adopt control measures or not will impact on the spread of the disease to their neighbours, for successful control a co-ordinated control response is needed. Epanchin-Niell and Wilen (2015) consider the impact of local cooperative and coordinated control agreements to manage invasions in a landscape with multiple independent identical landowners. They find that the level of co-operation needed to mitigate damage caused by disease depends on the cost of controls. In particular, their work shows that strategic interactions between independent landowners can be important to ensure a successful control response at the landscape scale. An interesting extension to the work presented here would be to incorporate strategic interactions between two forest managers with differing objectives, so as to investigate how the decision or whether or not to adopt control impacts the timing of control adoption for a neighbouring forest manager with different objectives.

In this article we have considered a control measure that reduces that rate at which a tree disease spreads, which could, for example, be the spraying of fungicides or pesticides that reduce the susceptibility of trees, removal of weeds to increase the vigour of the trees and reduce humidity or movement restrictions that reduce the chance of infected material being bought in from elsewhere. However, other control measures involve the removal of infected material, or treating currently infected trees, rather than altering the transmission rate parameter. The model presented here could be extended to explore how the optimal timing of control depends on the way in which a control measure alters disease spread. The model could also be extended to consider the optimal timing of control over multiple time periods, which would allow us to take into account that some non-timber benefits accumulate over multiple rotations.

This article represents the first attempt to investigate how the interaction between future uncertainty in disease spread and forest management objectives affect when (if ever) it is optimal for an individual forest manager to adopt control measures to reduce damage due to invasive pathogens or pests. We have shown that managers will adopt control at different times depending on the management objectives of the forest, specifically what value is placed on the various benefits from the forest, and the relative impact that disease has on these different benefits. Therefore, policy makers need to take into account the range of different management objectives to ensure a uniformed approach to disease control at the landscape scale.

Tables

Table 1: Parameter values used in numerical simulations.

Model Parameter	Description	Base case (Range)
β	Initial infection transmission rate (i.e. transmission rate when no control deployed)	0.15
ω	Reduction in infection transmission rate as a result of adopting control measures	0.1
F	Magnitude of fluctuations in infection transmission rate (uncertainty constant)	2.8
b	Net value per hectare of annual non-timber benefits	${}^{1}/_{2}\, re^{-rT}/(1-e^{-rT})$ (0 for timber manager, $re^{-rT}/(1-e^{-rT})$ for non-timber manager)
p	Net return per hectare from timber sold	${}^{1}/_{2}$ (1 for timber manager, 0 for non-timber manager)
φ	Factor by which non-timber benefits are reduced as a result of infection	0.7 (in Figure 1), 0.4 (in Figure 5) (0, 1)
ρ	Factor by which timber benefits are reduced by as a result of infection	0.7 (in Figure 1) (0, 1)
K_A	Fixed cost of control measures	0.001 ([0, 0.001])
m_A	Yearly maintenance cost of control measures	0.0001 ([0, 0.0001])
α	Proportion of initial sunk cost that is	0

	recouped upon cancelling control measures	
r	Risk-free interest rate	0.03
T	Time period over which to consider the value of the forest	80 years

Table 2*:* Lower and upper adoption and cancelation thresholds for baseline model, with different levels of uncertainty.

Uncertainty constant (F)	Lower adoption threshold (I_A^L)	Upper adoption threshold (I_A^U)	Lower cancellation threshold (I_C^L)	Upper cancellation threshold (I_C^U)
0	0.01	0.684	0	0.89
2.8	0.028	0.534	0.004	0.732

Table 3: Impact of different combinations of reductions to timber (ρ) and non-timber (φ) benefits on the optimal control strategy for a timber and non-timber manager, along with examples of diseases for which these combinations of ρ and φ arise.

Reduction in timber benefits	Reduction in non-timber benefits	Control strategy for timber manager	Control strategy for non-timber manager	Disease example
High ($\rho = 0.1$)	High ($\varphi = 0.1$)	Control when proportion of area infected within region	Control when proportion of area infected within region	Ash dieback

		[0.004, 0.8]	[0.01, 0.88]	
Low ($\rho = 0.9$)	Low ($\varphi = 0.9$)	Never adopt control	Never adopt control	
High ($\rho = 0.1$)	Low ($\varphi = 0.9$)	Control when proportion of area infected within region [0.004, 0.8]	Never adopt control	Dothistroma
Low ($\rho = 0.9$)	High ($\varphi = 0.1$)	Never adopt control	Control when proportion of area infected within region [0.01, 0.88]	Oak Processionary moth

Figures

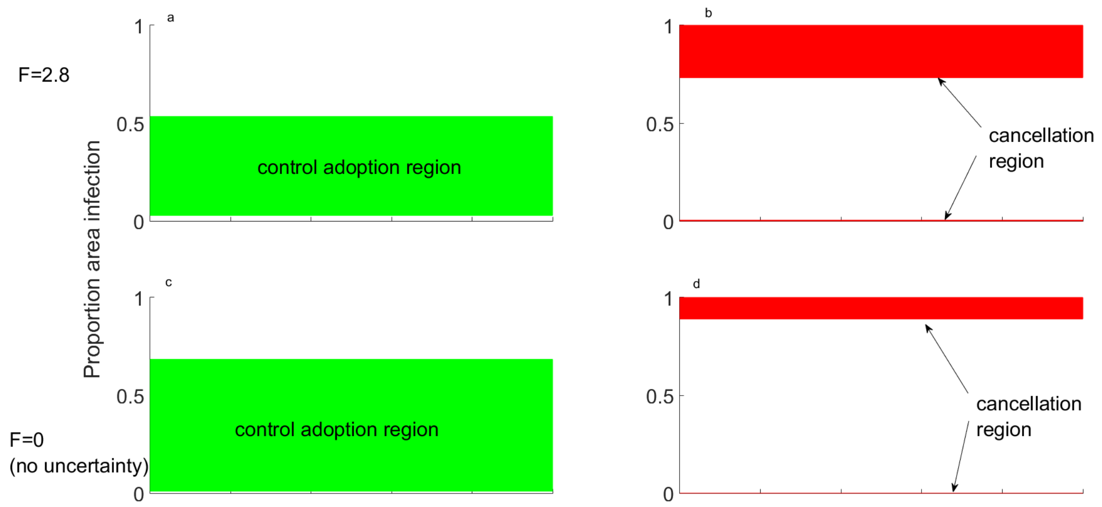

Figure 1: Adoption (a and c) and cancellation regions (b and d) when there is no uncertainty (c and d) and when uncertainty is included (a and b) in the decision problem (F=2.8). Parameter values used for simulation are given in Table 1. Coloured regions show where control should be adopted/cancelled immediately, while white regions show where the manager should wait and see.

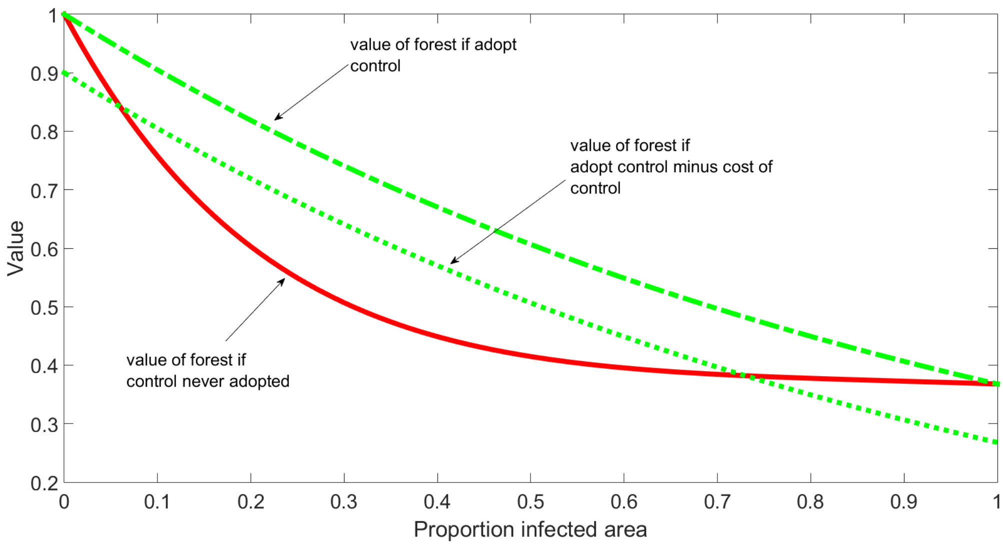

Figure 2: Stylised value of the forest if you adopt control and never cancel (green dot-dashed line) and value of the forest if control is never adopted (red solid line). Note that these are the value functions in the absence of any option values. The dotted green line indicates the value of the forest if you adopt control and never cancel minus the cost of control.

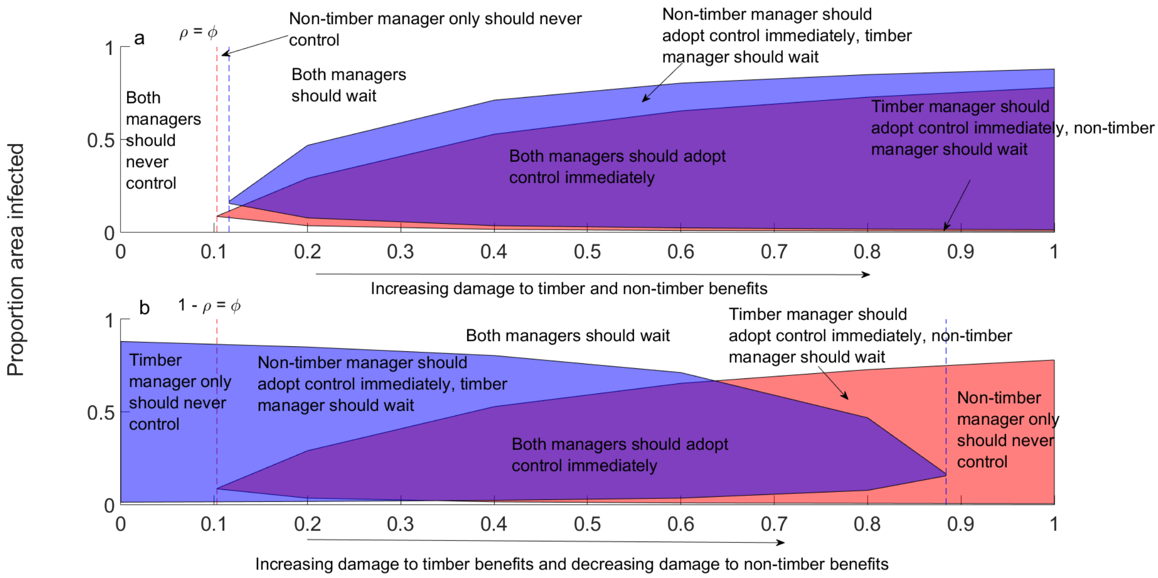

Figure 3: Adoption regions for timber-objective manager (red) and non-timber-objective manager (blue) when (a) the reduction in timber benefits equals the reduction in non-timber benefits ($1 - \rho = 1 - \varphi$) and when (b) the reduction in timber benefits is not equal to the reduction in non-timber benefits ($1 - \rho = \varphi$). Dashed lines show the switch points when moving from a region where the adoption thresholds exist to one where control should never be adopted.

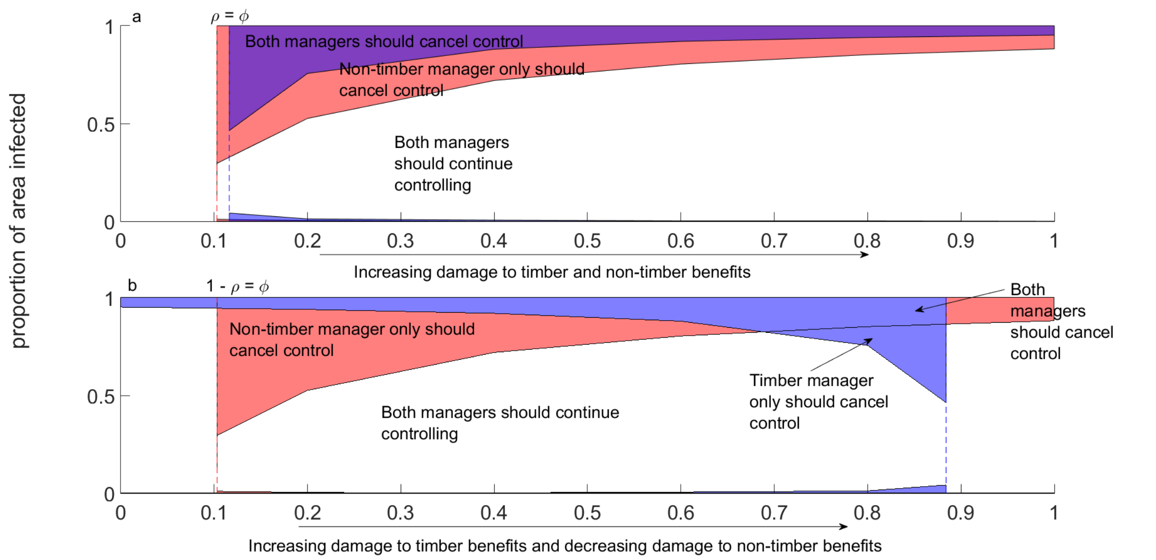

Figure 4: Cancellation regions for timber-objective manager (red) and non-timber-objective manager (blue) when (a) the reduction in timber benefits equals the reduction in non-timber

benefits ($1-\rho=1-\varphi$) and when (b) the reduction in timber benefits is not equal to the reduction in non-timber benefits ($1-\rho=\varphi$). Dashed lines show the switch points when moving from a region where the cancellation thresholds exists to one where control should never be cancelled.

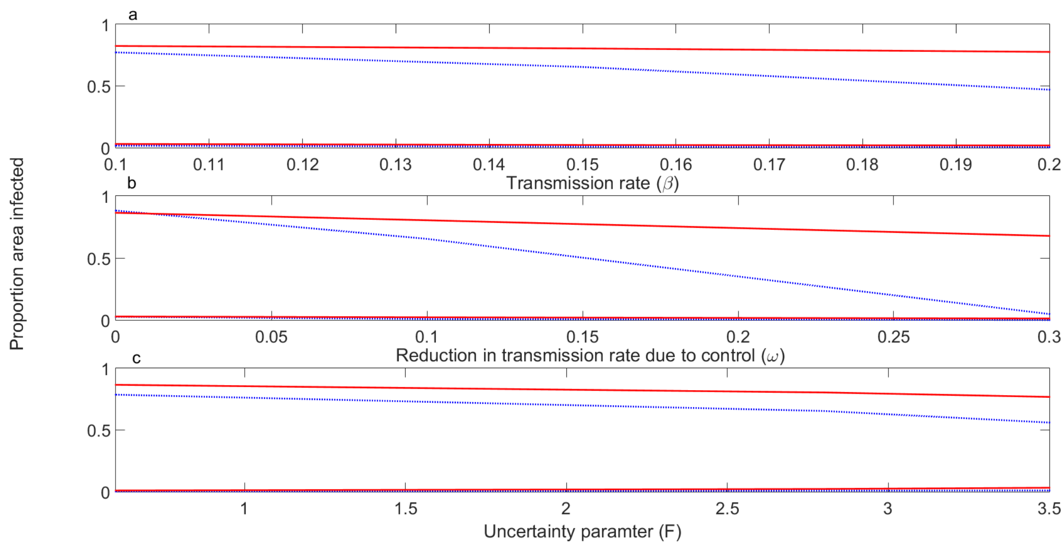

Figure 5: Upper and lower adoption thresholds for timber manager (blue dotted lines) and non-timber manager (red solid lines) as a function of: (a) the transmission rate β, (b) the reduction in transmission rate due to control ω and (c) the uncertainty parameter F. The reduction to timber and non-timber is assumed to be fixed, $\rho=\varphi=0.4$.

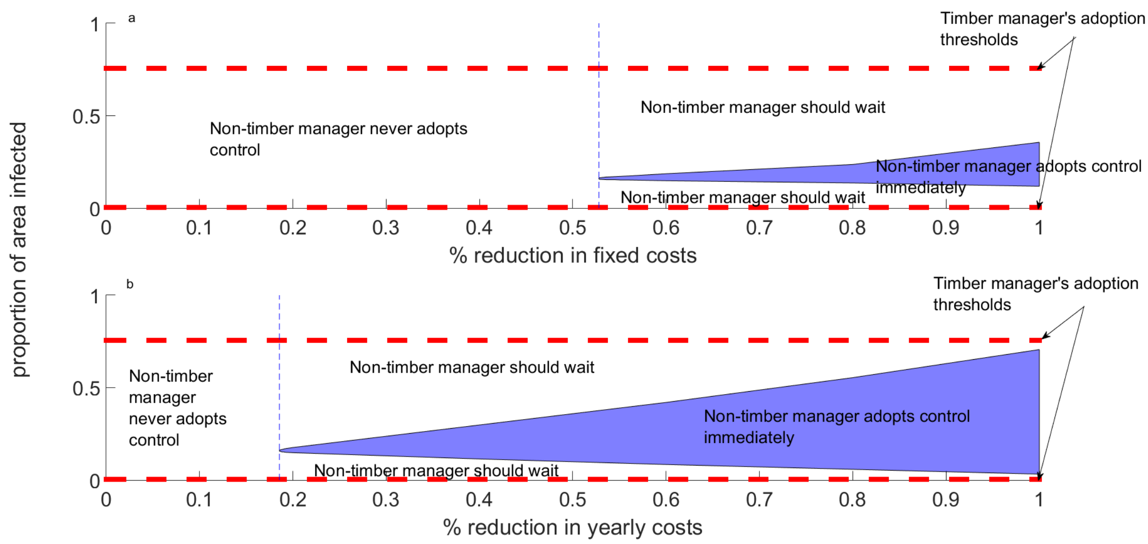

Figure 6: Impact of reducing fixed costs (a) and non-fixed costs (b) on the adoption region (blue shaded area) for the non-timber-objective manager, when the reduction in timber benefits is high ($\rho = 0.1$) and the reduction in non-timber benefits is low ($\varphi = 0.9$). The dashed red lines show the adoption thresholds for the timber manager when adoption costs are 0.001 and yearly maintenance costs are 0.0001.

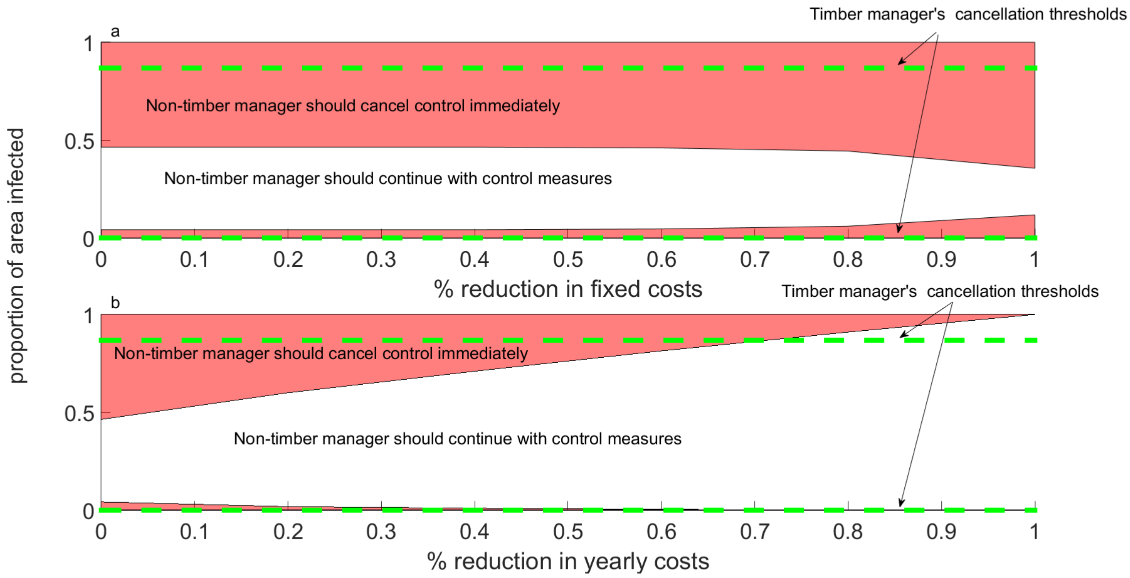

Figure 7: Impact of reducing fixed costs (a) and non-fixed costs (b) on the cancellation region (red shaded area) for the non-timber-objective manager, when the reduction in timber benefits is high ($\rho = 0.1$) and the reduction in non-timber benefits is low ($\varphi = 0.9$). The dashed green lines show the cancellation thresholds for the timber manager when adoption costs are 0.001 and yearly maintenance costs are 0.0001.

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Appendix

We solve the free boundary problem given by equations (10) and (11) along with the boundary conditions given by equations (6) to (9), (12) and (13) in MATLAB using the Euler method (Wilmott, Howison, and Dewynne 1995). To ensure numerical convergence, the time step used for numerical simulation, dt, must satisfy the following condition,

$$dt < \frac{1}{2(\sigma \times dI)^2},$$

where dI is the mesh size for the infected area variable (I) that is used in simulation (we take $dI = 0.002$ in all simulations). Since this is a free-boundary problem, in order to solve the problem numerically, an upper boundary condition must be stipulated in the case where $I = 1$ as well as a lower boundary condition in the case where $I = 0$. Therefore, we assume that ${\partial^2 W_A}/{\partial \mathrm{I}^2} = {\partial^2 W_C}/{\partial \mathrm{I}^2} = 0$ at $I = 1, 0$ which is used to obtain the finite difference scheme at the upper boundary (Insley 2002).